A THEORY OF QUANTUM COMMUNICATIONS

By

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

December, 1963

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to the former chairman of his supervisory committee, Dr. W.W. Peterson, to the current chairman, Dr. A.H. Wing, and to Dr. B.S. Thomas for their counsel and encouragement.

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Abstract of Dissertation Presented to the Graduate Council
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

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December, 1963

Chairman: Dr. A.H. Wing
Major Department: Electrical Engineering

The purpose of this thesis is to formulate a model for a bandlimited quantum communications channel and to investigate the channel capacity of such a channel in the absence of external and thermal noise.

The channel capacity of a quantum channel is affected by quantization in two important ways. The mere fact of quantization itself, i.e., that a quantized signal is restricted to a finite number of states which it can assume, limits the ability of the signal to contain information. The other factor which degrades the performance of a channel is that every bandlimited signal contains an irreducible uncertainty which is due to the uncertainty principle. This thesis represents an attempt to construct a reasonable model which takes these two factors into consideration.

Every signal in the proposed model is assumed to be associated with a wave function. All of the observable properties of a signal can be found from its wave function, and therefore, all of the information which a signal can contain is embodied in its wave function. Information is extracted from the signal by obtaining estimates of some or all of the properties of its wave function. The wave function of every signal, bandlimited to a bandwidth W and approximately timelimited to a time interval T, can be expressed as an expansion of 2TW + 1 prolate spheroidal wave functions.

The prolate spheroidal wave functions are eigenfunctions of a certain integral equation. The integral in this equation can be represented as an operator which is Hermitian, and corresponding to this operator is an observable of the signal. In this model, a receiver which measures this observable for each received photon in a signal can form an estimate of the wave function of the signal. The received signal can be represented by a set of 2TW + 1 numbers which give the distribution of the received photons into 2TW + 1 different states. In the absence of external and thermal noise each of these numbers is a random variable with a Poisson distribution.

Bounds for the channel capacity of a bandlimited quantum channel in the absence of external or thermal noise are found. The upper bound is derived from a procedure commonly used in statistical mechanics. The lower bound is derived using the Gaussian approximation of a Poisson distribution. It is shown how the quantum channel model formulated herein lends itself to

analyses which include the effects of external and thermal noise. Channel capacity computations can be made for these cases if the statistics of this noise is known.

The quantum nature of electromagnetic signals is discussed briefly in general terms.

CHAPTER 1

QUANTUM EFFECTS

Introduction

Shannon's famous formula [1] for the channel capacity C of a continuous channel of bandwidth W, signal power S, and white thermal noise power N is

$$C = W \log \frac{S + N}{N} .$$

This formula applies to a model of a communications channel called the continuous channel. In this model the signal s(t) and noise n(t) are represented by continuous functions of time. The quantized nature of energy and the Heisenberg uncertainty principle assure us that the continuous channel does not correctly represent an electromagnetic communications channel because in such a channel both signal and noise are discrete and their measurement somewhat uncertain.

Even though we assume that the electromagnetic field at the receiver is continuous, the problem of determining the field by natural means involves the quantization of nature and the Heisenberg uncertainty principle as it applies to the canonically conjugate variables, time and energy. Since we cannot identify our received message with infinite resolution, we cannot attain an infinite rate of information identification. There is, therefore, an irreducible uncertainty associated with

every electromagnetic signal, and it is the purpose of this study to investigate the nature of this uncertainty and to determine its effect on the communication of information.

We limit ourselves to the study of bandlimited signals. We are not concerned with any contamination of the signal due to thermal or external noise in the medium of transmission. However, it is shown how the procedures and results of this study may be extended to cover the case where thermal and external noises are present. The only noise considered is the quantum noise which arises from the fact that, since nature is quantized, we cannot have a physical disturbance which has precisely the characteristics of a continuous bandlimited signal. Further, the Heisenberg uncertainty principle imposes an additional uncertainty in the detection of the signal.

The development of the laser has made possible the generation of phase coherent monochromatic light. The photons of such a light signal all have the same energy. Modulation may be thought of as a process which changes the energy distribution of a monochromatic signal whose photons all have the same energy to a signal whose photons may have different energies. For example, if a monochromatic electromagnetic signal is modulated with a sinusoid, then the energy of the modulated signal is known to exist at the carrier frequency and at the two side-band frequencies. The modulation process has redistributed the energy of the photons so that some will be found to exist at all three frequencies. It is postulated in this study that every signal of time duration T and bandwidth W consists of photons which may be in

2TW + 1 different states. Any two signals which differ from each other do so simply because the distribution of their photons into these 2TW + 1 states differ.

The natural language for such a study is the formalism of quantum mechanics. Gordon [2] and Stern [3] have applied the principles of statistical mechanics to the channel capacity problem with considerable success. The results found here frequently agree qualitatively with theirs. Gordon's paper contains a good collection of references of recent work in this area.

Quantum concepts affect communication signals in two important ways. The first of these is the mere fact of quantization itself. The second is derived from the Heisenberg uncertainty principle, a consequence of which is that measurements performed on a bandlimited signal contain an irreducible uncertainty.

To illustrate the effect of quantization, consider a signal which may assume a value from zero to 99. If the amplitude of the signal is continuous and may assume any value from zero to 99, then, in the absence of noise, the signal contains an infinite amount of information. If the signal is quantized so that it can assume only the integer values 0, 1, 2, ..., 99, then the signal can contain only log 100 units of information. If the signal were quantized to levels which were spaced so that it could assume only even integers, then the information content of the signal would be reduced to log 50 units of information. Thus, the mere fact of quantization affects the channel capacity of a system.

The Heisenberg uncertainty principle states that any two canonically conjugate variables, such as time and energy, cannot simultaneously be measured with an arbitrarily small error. The energy in a signal exists in packets or photons and by specifying that a signal is bandlimited, we fix the energy of the photons to lie in a certain interval with probability one. Therefore, since the uncertainty in the energy is not zero, the uncertainty in the time cannot be zero. This concept is discussed in detail later, for the major topic of this paper is to investigate and clarify the nature of this uncertainty.

The work in this study leans heavily on the work of Slepian, Landau and Pollak [4, 5, 6] on prolate spheroidal wave functions. These functions have certain properties which make them ideally suited for use in the expansion of bandlimited functions.

Outline

This thesis consists of six chapters and three appendices.

Chapter 1, the present chapter, contains an introduction and an outline of this thesis.

Chapter 2 introduces the model of the communications system which we postulate. It discusses briefly the concept of the wave function of a signal and shows how the wave function can be expanded in terms of the eigenfunctions of a certain integral equation.

Chapter 3 contains a brief discussion of the concept of an observable and of how an observable is represented in quantum mechanics. The nature of the wave function and its relationship to observables are investigated. Vector notation is introduced.

In Chapter 4 a quantum channel is introduced which expresses the properties of the communications channel of Chapter 2 in the language of the quantum mechanics. The number of photons observed in each degree of freedom of the channel is shown to have a Poisson distribution. An upper and lower bound on the channel capacity of the quantum channel is found and illustrated graphically for several cases of interest.

In Chapter 5 the inherent ambiguity of signals is discussed in general terms. It is suggested that the concept of signals as continuous functions of time, a concept which in the past seemed indigenous to communications theory, may lead to erroneous results in some cases.

Chapter 6 contains a summary of conclusions.

CHAPTER 2

WAVE FUNCTIONS AND PROLATE SPHEROIDAL FUNCTIONS

A Communications Channel

The communications channel model used in this analysis is the one shown in Figure 1. The transmitter transmits a signal consisting of a large number of photons. The receiver receives N of these photons, analyzes them, and attempts to determine what signal was transmitted. From Figure 1 it is clear that the energy propagated in the y and z directions is of no interest because only energy propagated in the negative x direction reaches the receiver. This thesis is valid, however, even if all of the transmitted energy is analyzed by the receiver. The negative x direction is used as the direction of propagation simply because it leads to a desirable sign convention. The signals of the model postulated are polarized in only one direction and are not distorted by external or thermal noise.

Wave Functions

In quantum mechanics a wave function is usually defined to be a solution of a Schrödinger wave equation. The propagation of light is not governed by a Schrödinger wave equation, and therefore, light quanta do not have wave functions in the above strict definition of the term.

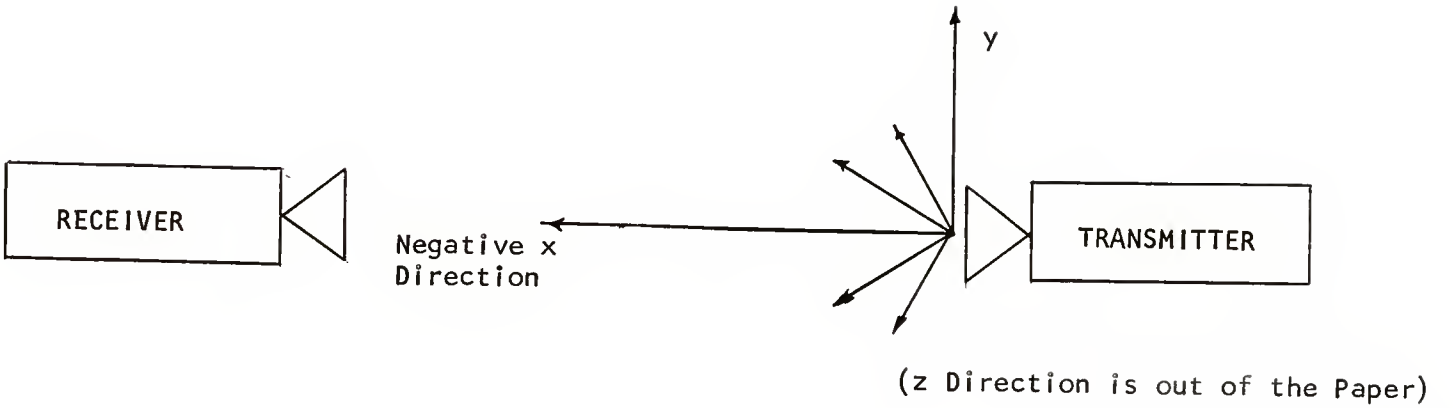

FIGURE 1 -- Model of a Communications System

For the purposes of this thesis, however, we will assign to every signal or beam of photons a function $\Psi(x, y, z, t)$, which we will call a wave function even though it is not a solution of any Schrödinger wave equation. This function is called a wave function mainly because it possesses many of the properties of the more conventional wave function. The receiver observes the signal as an ensemble of photons, and the wave function of this ensemble is $\Psi(x, y, z, t)$. Since the y and z variations are not detectable by the receiver, the wave function of interest is the one dimensional one $\Psi(x,t)$. Only one polarization is being considered in this analysis.

It is not the purpose of this study to discuss quantum communications in terms of quantum field theory, although such an endeavor seems worthy of investigation. We assume that the transmitter is designed so that it will emit a one photon state of the quantized electromagnetic field. The position representation of quantum mechanical states is the usual magnetic vector potential $\vec{A}(\vec{r}, t)$, and the fields derived from this potential obey Maxwell's equations and the second order wave equation. In this study we are taking into account the effect of the uncertainty principle by realizing that we cannot know that we have a one photon state and, at the same time, know the position of this photon. Strictly speaking, there is no wave function in quantum field thoery, but the function $\vec{A}(\vec{r}, t)$ is analogous to the wave function Ψ used in this paper and $|\vec{A}(\vec{r}, t)|^2$ gives the probability density for finding photons at $\vec{r}$ and t.

Undispersed Propagation

Propagation of a photon in free space is an example of undispersed propagation [7]. This is true because the velocity of propagation c is independent of the frequency of the photon. Thus, if a square pulse of photons is created at the transmitter, the pulse retains its shape as it propagates, and it is received as a square pulse. The analyses which follow can be generalized to apply to any signal which is undispersed.

The time independent wave function is obtained from $\Psi(x, t)$ by setting $t = 0$ and getting $\Psi(x, 0)$. $\Psi(x, 0)$, or simply $\Psi(x)$, and its Fourier transform $a(k)$ are related by

$$\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(k)\, e^{jkx}\, dk \quad , \tag{2.1}$$

$$a(k) = \int_{-\infty}^{\infty} \Psi(x)\, e^{-jkx} dx \quad .$$

$a(k)$ is sometimes written as $a(p)$ where $p = \hbar k$ and is called the momentum wave function. k is the wave number. $\Psi(x)$ and $a(k)$ are, in general, complex. Usually, it is the custom in the literature of physics to define the relationship between $\Psi(x)$ and $a(k)$ in terms of the symmetric transform pair which has a factor $1/\sqrt{2\pi}$ in front of both integrals of (2.1). We define our transform as in (2.1) because it allows us to use the prolate spheroidal wave functions unaltered from the literature [4] on this subject.

The physical interpretation of the wave function $\Psi(x)$ of a non-relativistic particle is that if the particle has wave function $\Psi(x)$, the probability density that it will be found at the position x is $|\Psi(x)|^2$ [8, 9]. Similarly, the probability density function that the wave number of the particle is found to be k is $|a(k)|^2$. The wave function for a photon signal has a similar interpretation. $|\Psi(x)|^2$ is the probability density function for finding photons at x. Similarly, the probability density function for finding photons with wave number k is $|\Psi(k)|^2$.

If the propagation is undispersed, then the relationship between x and t is $x = x_0 + ct$ and $\Psi(x, 0) = \Psi(x_0 + ct, 0)$. We have assumed propagation in the negative x direction, which conforms to the convention of Figure 1. By a correct choice of axis we can let $x_0 = 0$ and speak of the wave function as $\Psi(ct, 0)$ or simply $\Psi(ct)$. Making the change of variables $x = ct$ and $k = \omega/c$ in (2.1) gives

$$\Psi(ct) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a\left(\frac{\omega}{c}\right) e^{j\omega t} \frac{d\omega}{c} ,$$

$$a\left(\frac{\omega}{c}\right) = \int_{-\infty}^{\infty} \Psi(ct) e^{-j\omega t} dct .$$

Normalizing these so that $|\Psi(t)|^2$ and $|a(\omega)|^2$ represent probability density functions gives

$$\Psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega)\ e^{j\omega t}\ d\omega,$$

(2.2)

$$a(\omega) = \int_{-\infty}^{\infty} \Psi(t)\ e^{-j\omega t} dt\ .$$

Thus, the probability density function for the detection of photons at time t is $|\Psi(t)|^2$. The probability density function that photons have frequency ω (or energy $\hbar\omega$) is $|a(\omega)|^2$. The integral of $|\Psi(t)|^2$ over a certain time interval is proportional to the number of photons detected in this time interval.

Bandlimited Representation in Terms of Eigenfunctions

The quantities of most interest to us in what follows are $\Psi(t)$ and $a(\omega)$, and we will refer to these as the wave function and the frequency wave function, respectively. It is expedient in quantum mechanics to represent $\Psi(t)$ in terms of a series of orthonormal functions, and this procedure will be followed here. The set of orthonormal functions to be used is the prolate spheroidal wave functions of Slepian, Landau, and Pollak [4, 5, 6]. Appendix A is devoted to a brief explanation of the origin and principal characteristics of these functions.

The spectrum of the prolate spheroidal wave functions is confined to the bandwidth $(-\Omega, \Omega)$, which is centered on $\omega = 0$. These functions are, therefore, video type functions, and the analyses in this chapter are concerned only with these video functions. The usual electromagnetic

communications signals are narrow band signals whose frequency spectra are zero outside of a narrow band of frequencies. Such signals are called bandpass signals. Appendix B shows how the concepts in this chapter are extended to cover the bandpass case.

The prolate spheroidal wave functions are a set of real functions of the real variable t which satisfy the following three equations:

$$\int_{-\infty}^{\infty} \Psi_i(t)\, \Psi_j(t)\, dt = \delta_{ij} \,, \tag{A-1}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_i(t)\, \Psi_j(t)\, dt = \lambda_i\, \delta_{ij} \,, \tag{A-2}$$

$$\lambda_i \Psi_i(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega\,(t - s)}{\pi(t - s)}\, \Psi_i(s)\, ds \,, \tag{A-3}$$

where the δ_{ij} is the Kronecker delta function. The λ_i and $\Psi_i(t)$ are the eigenvalues and eigenfunctions, respectively, of the integral equation (A-3). Every $\Psi_i(t)$ is bandlimited to the band $(-\Omega,\Omega)$, i.e., the Fourier transform of every prolate spheroidal wave function is zero outside the interval $(-\Omega,\Omega)$.

Expressing a wave function $\Psi(t)$ as a series of these eigenfunctions gives

$$\Psi(t) = a_0\Psi_0(t) + a_1\Psi_1(t) + a_2\Psi_2(t) + \ldots \tag{2.3}$$

$$= \sum_i a_i \Psi_i(t) \,,$$

where the a_i may, in general, be complex. The $\Psi_i(t)$ are all real. The completeness of the set of prolate spheroidal wave functions assures us that every function limited to the band $(-\Omega,\Omega)$ can be expressed in a series of the form (2.3). We will deal only with $\Psi(t)$ which are normalized so that $|\Psi(t)|^2$ is a proper probability density function. This requires that

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \, dt = 1 \,, \tag{2.4}$$

and therefore from the orthonormality relation (A-1)

$$\sum_i |a_i|^2 = 1 \,. \tag{2.5}$$

A Theorem of Landau and Pollak

The great utility of this set of funtions is embodied in the following theorem stated by Landau and Pollak [6]. The theorem is slightly paraphrased for our use.

Theorem: Let $\Psi(t)$ be any function of time whose spectrum is zero outside the interval $(-\Omega,\Omega)$, and let $\Psi(t)$ be normalized so that

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \, dt = 1 \,.$$

If

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |\Psi(t)|^2 \, dt = 1 - \epsilon_T^2 \,, \tag{2.6}$$

then

$$(2.7) \qquad \underset{\{a_i\}}{\mathrm{INF}} \int_{-\infty}^{\infty} \left| \Psi(t) - \sum_{i=0}^{2TW} a_i \Psi_i(t) \right|^2 dt < 12\, \epsilon_T^2 ,$$

where the $\Psi(t)$ are the prolate spheroidal wave functions, and $W = \frac{\Omega}{2\pi}$.

Equation (2.7) states that a set of complex numbers (a_0, a_1, ..., a_{2TW}) can always be found such that the integral on the left side of (2.7) is less than $12\, \epsilon_T^2$.

According to the postulates of our model there is a one-to-one correspondence between the frequency of a photon and a frequency lying in the band $(-\Omega,\Omega)$. This means that the counterpart frequency of each photon lies inside the band $(-\Omega,\Omega)$ with probability one. Thus, $|a(\omega)|^2$ must be zero outside the interval $(-\Omega,\Omega)$. $\Psi(t)$ is, therefore, bandlimited to $(-\Omega,\Omega)$. The interpretation of $a(\omega)$ for negative ω will be discussed in the sequel.

One way to state that a signal is approximately limited to the time interval (-T/2, T/2) is to state that the probability is small that photons will be received outside that interval. A precise statement of this is equation (2.6), which states that if a signal has a wave function which satisfies (2.6), then the probability of receiving photons outside the interval (-T/2, T/2) is ϵ_T^2. Thus, if N photons are received, then all but about $N\epsilon_T^2$ will be received in the time interval (-T/2, T/2).

If ϵ_T^2 is small, then a signal consisting of a large number of photons is approximately timelimited to the interval (-T/2, T/2).

Equation (2.7) states that if a bandlimited function is approximately timelimited in accordance with equation (2.6), then such a function can be approximated in the mean square sense by a truncated series of 2TW + 1 prolate spheroidal wave functions. Thus, we can represent any such wave function by a series of the first 2TW + 1 prolate spheroidal wave functions

$$\Psi(t) = \sum_{i=0}^{2TW} a_i \Psi_i(t) \ , \tag{2.8}$$

and the mean square error of this representation can be made less than $12\epsilon_T^2$ with the proper choice of the a_i. It is shown in [4] that this representation becomes more exact in the interval (-T/2, T/2) as T gets larger.

CHAPTER 3

OBSERVABLES

Hermitian Operators

According to quantum mechanics [7, 8, 9], every physically measurable property of a signal corresponds to a Hermitian operator, such that the eigenvalues of the operator are the possible results of a measurement of this observable.

A Hermitian operator is an operator R which satisfies the equation

$$(\Psi, R\Phi) = (R\Psi, \Phi) \quad , \tag{3.1}$$

where (f, g) means the scalar or inner product and is defined by

$$(f, g) = \int_{-\infty}^{\infty} f^{*} g \, dt \quad .$$

Consider the integral operator of equation (A-3), namely

$$R = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi (t - s)} \{ \quad \} \, ds \quad . \tag{3.2}$$

The following equation shows that the eigenfunctions of the operator R are the prolate spheroidal wave functions and the eigenvalues are the corresponding set of numbers $\lambda_0, \lambda_1, \ldots$;

$$(3.3) \qquad R\,\Psi_i(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi\,(t - s)}\ \Psi_i(s)\ ds = \lambda_i \Psi_i(t)\ .$$

That the operator R is Hermitian in the space of prolate spheroidal wave functions is verified as follows. Let

$$\Psi(t) = \sum_i a_i \Psi_i(t)$$

and

$$\Phi(t) = \sum_i b_i \Psi_i(t)\ ,$$

then

$$(\Psi,\ R\Phi) = \left(\sum_i a_i \Psi_i,\ R \sum_i b_i \Psi_i\right)$$

$$= \left(\sum_i a_i \Psi_i,\ \sum_i b_i \lambda_i \Psi_i\right)$$

$$= \sum_i a^*_i b_i \lambda_i\ ,$$

which agrees with

$$(R\Psi,\ \Phi) = \left(R \sum_i a_i \Psi_i,\ \sum_i b_i \Psi_i\right)$$

$$= \left(\sum_i a_i \lambda_i \Psi_i,\ \sum_i b_i \Psi_i\right)$$

$$= \sum a^*_i b_i \lambda_i\ .$$

The theory of measurements further states that if we measure the observable R of a system whose wave function is

$$\Psi(t) = \sum_i a_i \Psi_i(t) \ ,$$

then the probability that the result is λ_0 is $|a_o|^2$ and, in general, the probability that the result is λ_k is $|a_k|^2$. Therefore, if we measure R for a large number N of photons, the expectation of the number of times we observe λ_k is $N|a_k|^2$ for $k = 0, 1, \ldots, 2TW$.

We have already suggested that the wave function of a signal contains all the information which a signal can have, and that the wave function of a signal describes it completely. The above analysis, therefore, suggests a model for a receiver. Suppose a receiver measured the observable R on N different photons and measured λ_0 n_0 times, λ_1 n_1 times, and in general, λ_k n_k times. The quantities n_0/N, n_1/N, ... would give estimates of the quantities $|a_0|^2$, $|a_1|^2$, ... This receiver is explained in greater detail in the next chapter.

The Dimensionality of the Wave Function

The general theory of quantum mechanics allows $\Psi(t)$ to be complex. Since the set $\Psi_i(t)$ consists of real functions only, this means that the coefficients a_i in (2.3), or (2.8) in the approximately timelimited case, are generally complex. It is the purpose of this section to show that a wave function representing undispersed wave motion can be uniquely determined by a set of 2TW+1 real numbers.

The sampling theorem states that, if a signal of bandwidth W is properly sampled at a rate of 2W samples per second, then knowledge of the sample values is sufficient to reconstruct the signal exactly. This implies that a signal limited to bandwidth W and time duration T has about 2TW independent variables or degrees of freedom. Examination of (2.8) shows that if the real and imaginary parts of each a_i were all independent, then the wave function representing a bandlimited signal would have about 4TW degrees of freedom. If the imaginary part of each a_i is determined by the real part, however, then there are only 2TW degrees of freedom, as the past has led us to believe.

The determination of the number of degrees of freedom for a wave function $\Psi(t)$ is embodied in the interpretation of the frequency wave function $a(\omega)$ for negative frequencies. $a(\omega)$ is derived from $a(k)$ or $a(p)$, and $a(p)$ does have meaning for negative p. $|a(p)|^2$ for negative p is simply the probability density function for a measurement of momentum in a direction in which the momentum p is considered negative. The equivalent interpretation for $a(\omega)$ for negative ω is that $|a(\omega)|^2$ for negative ω is the density function for a measurement of the frequency of a particle or photon traveling in the direction from receiver to transmitter. In a practical communications system the signal of interest travels from transmitter to receiver and $a(\omega)$ is zero for negative ω. With this in mind, let us define a wave function $\Psi(t)$ whose spectrum $a(\omega)$ is zero outside the interval $(0,\Omega)$. Such a function can be represented as

$$\Psi(t) = \Phi(t) + j\hat{\Phi}(t) \quad , \tag{3.4}$$

where $\Phi(t)$ is a real function whose Fourier transform is $\alpha(\omega)$ defined by

$$\alpha(\omega) = 1/2\ a(\omega) \qquad \omega > 0$$
$$= 0 \qquad \omega = 0$$
$$= 1/2\ a^*(-\omega) \qquad \omega < 0 \quad . \tag{3.5}$$

$\hat{\Phi}(t)$ is also a real function whose Fourier transform $\alpha'(\omega)$ is given by

$$\alpha'(\omega) = -\ 1/2\ j\ a(\omega) \qquad \omega > 0$$
$$= 0 \qquad \omega = 0$$
$$= 1/2\ j\ a^*(-\omega) \qquad \omega < 0 \quad . \tag{3.6}$$

The function $\hat{\Phi}(t)$ is the Hilbert transform of $\Phi(t)$, and the relationship between $\Phi(t)$ and $\hat{\Phi}(t)$ is given by

$$\Phi(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{\Phi}(\tau)}{t-\tau}\, d\tau$$
$$\hat{\Phi}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Phi(\tau)}{t-\tau}\, d\tau \quad . \tag{3.7}$$

The integrals are taken as Cauchy principal values. A good explanation of the use of Hilbert transforms in communications theory is given by Dugundji [10]. (Dugundji defines his transforms as the negative of the way they are defined classically, as in Titchmarsh [11] for example. We use Dugundji's sign convention mainly because we can use and refer to his work on spectrum analysis without altering the signs of his results. The results reported in Dugundji's paper are easily derived from Titchmarsh's analysis of Hilbert transforms.)

We see from (3.4) that the real part of the wave function $\Psi(t)$ determines the imaginary part through the Hilbert transform relation or through the relationships of their spectra. Now the spectrum of $\phi(t)$ is limited to the interval $(-\Omega,\Omega)$, and we have already shown that such a function can be approximated in the time interval $(-T/2, T/2)$ by the series

$$\phi(t) = \sum_{i=0}^{2TW} b_i \Psi_i(t) \quad . \tag{3.8}$$

Similarly, $\hat{\phi}(t)$ can be expressed as

$$\hat{\phi}(t) = \sum_{i=0}^{2TW} d_i \Psi_i(t) \quad . \tag{3.9}$$

An examination of the functions $\alpha(\omega)$ and $\alpha'(\omega)$ shows that $\phi(t)$ and $\hat{\phi}(t)$ are both real; therefore, the coefficients b_i and d_i are both real and $\Psi(t)$ can be written as

$$\Psi(t) = \sum_{i=0}^{2TW} a_i \Psi_i(t) = \sum_{i=0}^{2TW} (b_i + j d_i)\, \Psi_i(t) = \sum_{i=0}^{2TW} A_i e^{j\theta_i}\, \Psi_i(t) \quad . \tag{3.10}$$

The fact that $\Psi(t)$ is normalized in accordance with equation (2.4) leads to

$$\sum_{i=0}^{2TW} (b_i^2 + d_i^2) = \sum_{i=0}^{2TW} A_i^2 = 1 \quad . \tag{3.11}$$

Titchmarsh [11] shows that, since $\Phi(t)$ and $\hat{\Phi}(t)$ are Hilbert transforms,

$$\int_{-\infty}^{\infty} |\Phi(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{\Phi}(t)|^2 \, dt \, . \tag{3.12}$$

Therefore,

$$\sum_{i=0}^{2TW} b_i^2 = \sum_{i=0}^{2TW} d_i^2 = \frac{1}{2} \, . \tag{3.13}$$

Since $\Phi(t)$ is real and is normalized by equation (3.13), it has only 2TW degrees of freedom, and consequently $\Psi(t)$, which is determined once $\Phi(t)$ is known, has only 2TW degrees of freedom. Thus, an ordered set of 2TW real numbers $(b_0, b_1, b_2, \ldots, b_{2TW})$ with the restriction of (3.13) is sufficient to determine $\Psi(t)$ uniquely.

Vector Notation

It is convenient to represent wave functions and operators in terms of vectors in a linear vector space. (Most quantum mechanics texts treat this topic with some detail [12].) In this scheme

$$\Psi(t) = \sum_{i=0}^{2TW} a_i \Psi_i(t) \tag{2.8}$$

is represented in vector notation as

$$\Psi = (a_0, a_1, \ldots, a_{2TW}) \, . \tag{3.14}$$

In these two equations there are only 2TW + 1 coefficients a_i, because we are representing bandlimited functions which are timelimited in the sense of Landau and Pollak's theorem. In the general case the number of coefficients a_i is unlimited.

The basis vectors for the vector space of all bandlimited, approximately timelimited wave functions are, of course, the 2TW + 1 prolate spheroidal wave functions, and these are represented in the vector space as

$$\begin{aligned} \Psi_0 &= (1, 0, 0, \ldots, 0), \\ \Psi_1 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \Psi_{2TW} &= (0, 0, 0, \ldots, 1) \quad . \end{aligned} \tag{3.15}$$

The operator R is represented by the Hermitian diagonal matrix

$$R = \begin{bmatrix} \lambda_0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & \lambda_{2TW} \end{bmatrix} . \tag{3.16}$$

Equation (3.1) is easily verified in the vector notation as is (3.3) which is written simply as

(3.17) $$R\Psi_i = \lambda_i \Psi_i \ .$$

When expressed as in equation (3.14) the Ψ is usually called a state vector. The Ψ_i of (3.15) are called eigenstates or eigenvectors.

Observables

Max Born [13] has lucidly stated the interpretation of the wave function or state vector somewhat differently than was stated at the beginning of this chapter. What follows in this paragraph is his statement slightly paraphrased to adapt it to our situation. To each physical quantity or observable belongs a real Hermitian operator R. The eigenfunctions Ψ_0, Ψ_1,... (or eigenstates in vector notation) correspond to the quantized or pure states, for which the operator takes on the eigenvalues λ_0, λ_1, ... Any wave function Ψ represents a state which can be considered as a mixture of pure states. The coefficients a_i of the expansion (2.8) or (3.14) determine the strength with which the quantum state Ψ_i occurs in the general state Ψ. The probability of finding the eigenvalue λ_i in a certain measurement is given by $|a_i|^2$ or A_i^2.

Thus, as far as the observable associated with the operator R is concerned, a signal in the general state Ψ consists of N photons each of which may be found in one of 2TW + 1 different states represented by the Ψ_i. In this study we have constructed a model in which information is derived from the observable of the signal represented by the operator R. The exact physical description of this observable is a subject for further study.

CHAPTER 4

QUANTUM CHANNEL MODEL

Messages and Signals

The discussion in the preceding chapters suggests the following model for a communications system. Some sets of positive real numbers $A = (A_0, A_1, \ldots, A_{2TW})$ represent messages with the restriction that

$$\sum_{i=0}^{2TW} A_i^2 = 1 \ . \tag{4.1}$$

Associated with each such message there is a bandlimited wave function $\Psi = (a_0, a_1, \ldots, a_{2TW})$ where $A_i = |a_i|$. In order to transmit the message $A = (A_0, A_1, \ldots, A_{2TW})$, the transmitter simply transmits energy with the corresponding wave function $\Psi = (a_0, a_1, \ldots, a_{2TW})$.

According to our model, a signal consists of photons each of which may be found in one of $2TW + 1$ states and the distribution of the photons into these states is determined by the quantity A. The function of the receiver is to determine the number of received photons in each state and, from this, to estimate the distribution function A of the signal.

The receiver receives N photons and performs the measurement R for each photon. The result of this measurement on each photon is one of the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{2TW}$. Thus, if the receiver measures λ_i n_i

times, this means that n_i of the N photons were found to be in state i. The receiver determines the set of numbers $n = (n_0, n_1, \ldots, n_{2TW})$ and this set represents the number of photons found in each of the 2TW + 1 states. The restriction on this set is

$$\sum_{i=0}^{2TW} n_i = N. \tag{4.2}$$

Since each n_i is a random variable with mean $A_i^2 N$ the quantity $+\sqrt{n_i/N}$ can be used to estimate A_i. On the basis of n, which we will call the received signal, the receiver attempts to estimate the transmitted message $A = (A_0, A_1, \ldots, A_{2TW})$.

The Poisson Distribution of the n_i

Each of the n_i are random variables. If N were a constant, then the set $(n_0, n_1, \ldots, n_{2TW})$ would have the following multinomial distribution

(4.3)

$$p(n_0 = \eta_0, \ldots, n_{2TW} = \eta_{2TW} | N) = \frac{N!}{\eta_0! \ldots \eta_{2TW}!} \left[(A_0^2)^{\eta_0} \ldots (A_{2TW}^2)^{\eta_{2TW}}\right].$$

However, the number of photons received in the time interval (-T/2, T/2) is a random variable with expectation value $\bar{N}$. If the time interval T

is sufficiently large, then the distribution of N is the Poisson distribution [14]

$$p(N) = \frac{e^{-\bar{N}}\,\bar{N}^{N}}{N} \quad , \tag{4.4}$$

where $\bar{N}$ is the expectation of N. We are generally interested in the case of large T, and for channel capacity considerations, in the case where T gets indefinitely large. Thus for sufficiently large T we can write (4.3) as:

$$p(n_0 = \eta_0, \ldots, n_{2TW} = \eta_{2TW}) = \left[\frac{N!}{\eta_0! \cdots \eta_{2TW}!}\right]\left[(A_0^2)^{\eta_0} \cdots (A_{2TW}^2)^{\eta_{2TW}}\right]\frac{e^{-\bar{N}}\,\bar{N}^{N}}{N!} \quad . \tag{4.5}$$

Making the substitution $N = \eta_0 + \eta_1 + \ldots + \eta_{2TW}$, and $\bar{N} = (A_0^2 + \ldots + A_{2TW}^2)\,\bar{N}$ gives

$$p(n_0 = \eta_0, \ldots, n_{2TW} = \eta_{2TW}) = \frac{(A_0^2\bar{N})^{\eta_0} \cdots (A_{2TW}^2\bar{N})^{\eta_{2TW}}}{\eta_0! \cdots \eta_{2TW}!}\, e^{(-A_0^2 - \cdots - A_{2TW}^2)\bar{N}}$$

$$= \prod_{i=0}^{2TW} \frac{(A_i^2\,\bar{N})^{\eta_i}\, e^{-A_i^2\,\bar{N}}}{\eta_i!} \tag{4.6}$$

$$= \prod_{i=0}^{2TW} \frac{(m_i)^{\eta_i}\, e^{-m_i}}{\eta_i!} \quad ,$$

where $m_i = A_i^2 \bar{N}$. Each term in the product (4.6) has the form of the Poisson distribution. Equation (4.6) clearly shows that the n_i are independent random variables, each with a Poisson distribution with mean $m_i = A_i^2 \bar{N}$.

Summarizing then, we have found that a bandlimited signal of time duration approximately T has 2TW independent degrees of freedom. Every message can be represented by a set of real positive numbers $(A_0, A_1, \ldots, A_{2TW})$ with the constraint of equation (4.1). For each of these degrees of freedom there is a number n_i representing the number of photons detected by the receiver in the state i. The n_i are each independent random variables with the Poisson distribution

$$p(n_i) = \frac{(m_i)^{n_i} e^{-m_i}}{n_i!} ,$$

where $m_i = A_i^2 \bar{N}$, and $\bar{N}$ is the expected value of the total number of photons received.

Figure 2 shows the elements of the model described in this section.

The input to the decision circuit is simply the set of numbers $n = (n_0, n_1, \ldots, n_{2TW})$. This means that there were n_i photons found to be in state i, for $i = 0, 1, \ldots, 2TW$. This input consists of a signal vector

(4.7) $$m = (m_0, m_1, \ldots, m_{2TW}), \text{ where } m_i = A_i^2 \bar{N},$$

plus a quantum noise vector

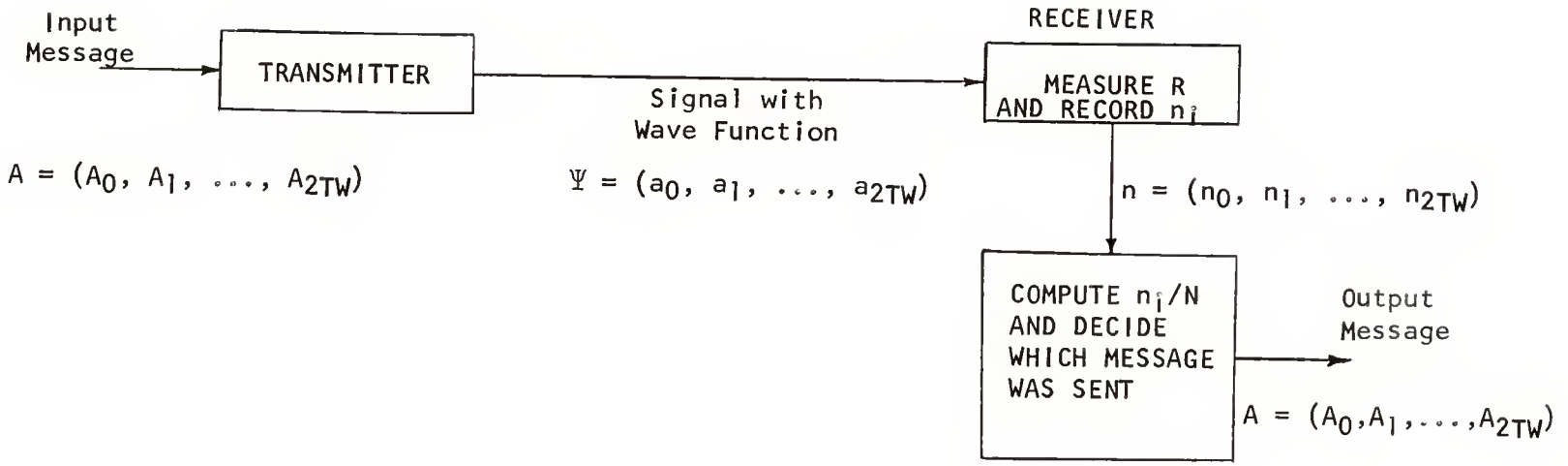

FIGURE 2 -- Block Diagram of the Quantum Channel

(4.8) $q = (q_0, q_1, \ldots, q_{2TW})$, where $q_i = n_i - m_i$.

where $q_i = n_i - m_i$. Thus, the input to the decision circuit can be represented as

(4.9) $n = m + q,$

where $n_i = m_i + q_i$.

Each n_i, being the number of times the receiver measured R to be λ_i, is an integer. The m_i are, in general, not integers. The quantum noise is not restricted to a particular set of quantized values since, although n_i is an integer, m_i is not.

As far as channel capacity computations are concerned, we can simplify the model of Figure 2 to the model shown in Figure 3. The simplified model uses m as its input signal or message. This is valid because m is completely determined by A. The vector q can represent any noise whose statistics are known, and the model proposed should be quite general. In what follows, however, we restrict our discussion to the case where only quantum noise is present.

Channel Capacity Upper Bound

Keeping in mind our somewhat unusual notation, namely that the signal is the vector $m = (m_0, m_1, \ldots, m_{2TW})$, the quantum noise is the vector $q = (q_0, q_1, \ldots, q_{2TW})$, and the signal plus quantum noise is the vector $n = m + q = (n_0, n_1, \ldots, n_{2TW})$, we will now derive an

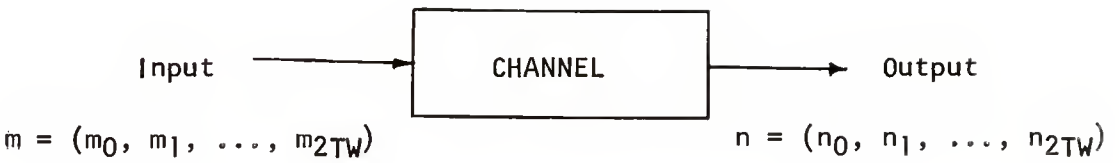

FIGURE 3 -- Simplified Block Diagram of the Quantum Channel

upper limit for the channel capacity of the model of Figure 4. The general formula for the channel capacity is given by Shannon [15] as

$$C = \lim_{T\to\infty} \frac{1}{T} \text{Max}(H(n) - H_m(n)), \tag{4.10}$$

where

$$H_n = -\sum_n p(n) \log p(n) \tag{4.11}$$

and

$$H_m(n) = -\sum_m p(m) \sum_n p(n|m) \log p(n|m) \quad . \tag{4.12}$$

$H(n)$ is the entropy of n, the signal plus quantum noise. $H_m(n)$ is the conditional entropy of n given the signal m.

The difficulty in getting an exact solution for equation (4.10) is a result of the fact that the signal and noise are not independent.

The channel capacity of any discrete channel can be no greater than the entropy rate of the received signal. We will, therefore, use this entropy rate of the received signal as an upper bound of the channel capacity. That is, we will use the fact that if the processes involved are all discrete as in this case, the C of equation (4.10) is always less than or equal to

$$H_R = \lim_{T\to\infty} \frac{1}{T} \text{Max}_{p(n)}\, H(n) \quad . \tag{4.13}$$

To make this computation, we use a procedure of statistical mechanics [16]. Let U be the number of distinguishable ways that N photons can be placed in 2TW + 1 compartments. Thus, if N photons were received, then U would be the number of different signals which could be received. In Figure 4 there are N photons, each represented by an X. 2TW lines are required to divide the N photons into 2TW + 1 compartments. The arrangement shown in Figure 4 represents the case where the zeroth compartment has three photons; therefore, $n_0 = 3$. The first compartment has one photon; therefore, $n_1 = 1$, et cetera. The vector represented by Figure 4 is, therefore, $n = (3, 1, 0, 3, \ldots, 2)$. The number of possible ways to arrange N + 2TW symbols is $(N + 2TW)!$ However, since there are two groups of indistinguishable symbols, i.e., the N X's and the 2TW lines, the number of distinguishable arrangements of these symbols is

$$U = \frac{(N + 2TW)!}{N!\ (2TW)!} \quad . \tag{4.14}$$

This is the same as the number of different vectors $n = (n_0, n_1, \ldots, n_{2TW})$ which can be formed with the constraint

$$\sum_{i=0}^{2TW} n_i = N.$$

FIGURE 4 -- Arrangement of N Photons into 2TW + 1 Compartments

The entropy of such a system is a maximum if each distinguishable event, i.e., each possible received signal, is equally likely, and this entropy is given by

$$H = \log U = \log (N + 2TW)! - \log N! - \log (2TW)! \quad . \tag{4.15}$$

Stirling's formula states for a large argument Z we can approximate log Z! as

$$\log Z! \simeq Z \log Z - Z \ .$$

Since we are interested in the entropy rate which is obtained from H as T and N approach infinity, we can use Stirling's formula in (4.15), giving

$$H = (N + 2TW) \log(N + 2TW) - (N + 2TW) - N \log N + N$$

$$- 2TW \log 2TW + 2TW$$

$$H = N \log \frac{N + 2TW}{N} + 2TW \log \frac{N + 2TW}{2TW} \quad . \tag{4.16}$$

Equation (4.16) states that if N photons are received in a time interval of length T and bandwidth W, then the amount of information (in bits, for example, if the logarithm base is 2) which can be received is given by H. The entropy rate is defined as

$$H_R = \lim_{T \to \infty} \frac{1}{T} H$$

$$= \lim_{T \to \infty} \left\{ \frac{N}{T} \log \frac{N + 2TW}{N} + 2W \log \frac{N + 2TW}{2TW} \right\} \ .$$

$$H_R = p \log \left(1 + \frac{2W}{p}\right) + 2W \log \left(\frac{p}{2W} + 1\right) , \tag{4.17}$$

where

$$p = \lim_{T \to \infty} \frac{N}{T} .$$

The quantity p is the average number of photons per second in the signal. The quantity H_R is the entropy rate of the received signal and may be measured in bits per second. It is the maximum entropy rate which the received signal may have. If there were no quantum noise, this H_R would be the channel capacity. Since there is quantum noise, we know that the channel capacity must be less than H_R, and we use H_R as an upper limit for the channel capacity C, i.e.,

$$C \leq H_R .$$

As a further simplification, we write (4.17) as

$$H_R = 2W \left\{ \frac{p}{2W} \log \left(1 + \frac{2W}{p}\right) + \log\left(\frac{p}{2W} + 1\right)\right\}. \tag{4.18}$$

Making the substitution d = p/2W gives

$$H_R = 2W\, C(d), \tag{4.19}$$

where

$$C(d) = d \log \left(1 + \frac{1}{d}\right) + \log (d + 1). \tag{4.20}$$

The quantity d is recognized as the average number of photons per second per degree of freedom. The quantity H_R is the channel capacity of a discrete channel when no quantum noise is present. Equations (4.18) and

(4.19) are similar to equations found by Stern [3] and by Gordon [2] using a different technique. Gordon noted that the first term in (4.18) was the dominant term when d is small and is, therefore, of fundamentally quantum origin. For large d the second term predominates. A plot of C(d) vs d is shown in Figure 5. We refer to C(d) as the normalized upper bound for the channel capacity.

Up to this point, we have dealt only with video signals, i.e., signals whose spectra $a(\omega)$ were either adjacent to zero or centered on zero. Both of these cases actually represent the same physical phenomenon, although different interpretations are required for each case. One case of great interest is that of a signal whose spectrum is bandlimited to higher frequencies which are not near zero, i.e., bandpass signals. That the results found in the video case can be extended to bandpass functions is the subject of Appendix B. We simply state here that the bound given by equation (4.19) applies to any signal whose spectrum is zero outside of some band of positive real frequencies of width W cycles per second.

If a narrow band signal is centered on the frequency ν, then we can say that any photon received over the channel has energy approximately $h\nu$. Since the number of photons received per second is p, the average received power P is

$$P = ph\nu = 2WdhN,$$

and

$$d = \frac{P}{2Wh\nu} \quad . \tag{4.21}$$

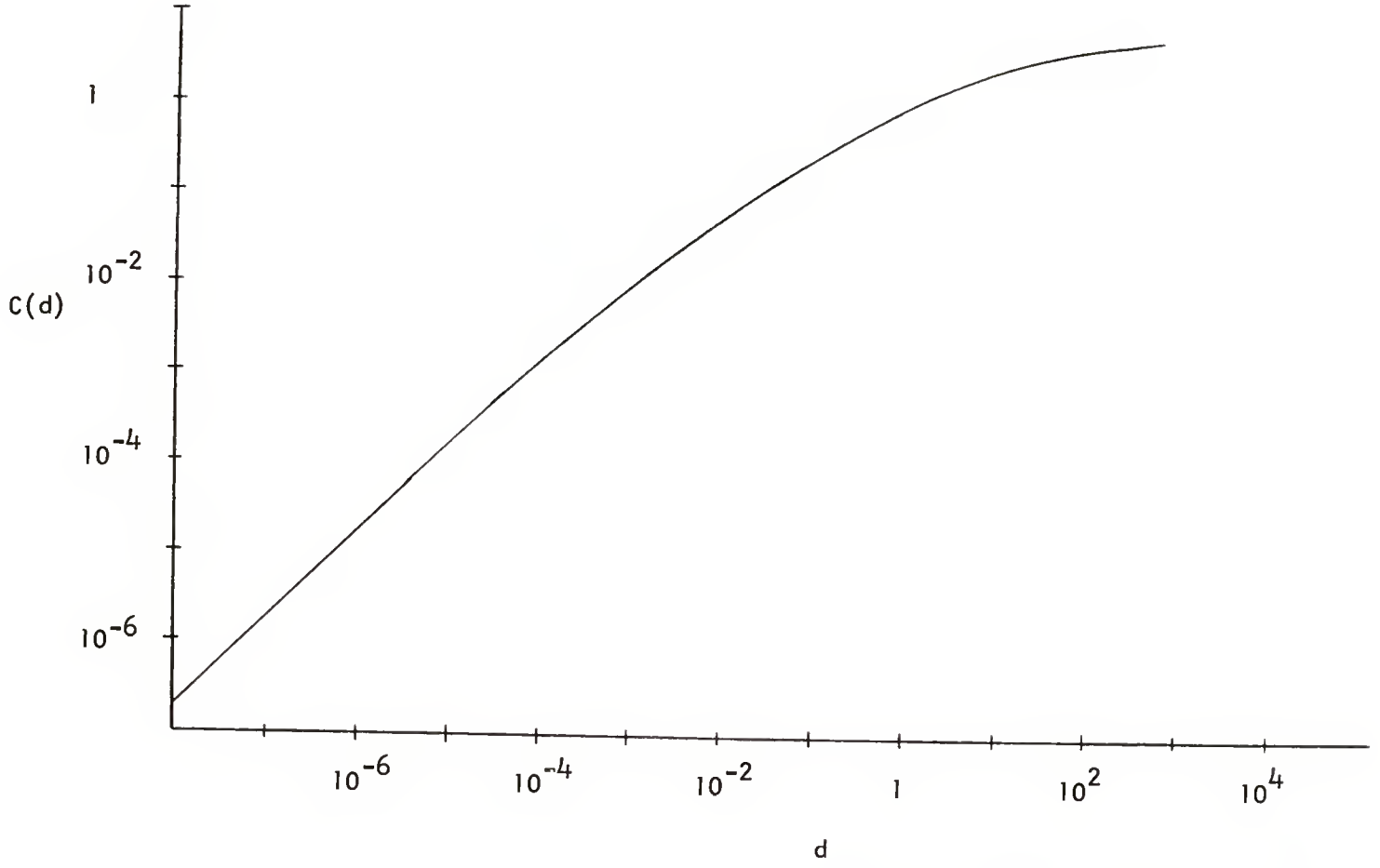

FIGURE 5 -- Normalized Upper Bound on Channel Capacity C(d) vs. d, the Average Number of Received Photons per Second per Degree of Freedom

Making this substitution in (4.18) and (4.19) gives an expression for the upper bound H_R of the channel capacity in terms of the average received power P. This expression is

$$(4.22) \qquad H_R = 2W \left\{ \frac{P}{2Wh\nu} \log \left(1 + \frac{2Wh\nu}{P}\right) + \log \left(\frac{P}{2Wh\nu} + 1\right)\right\}.$$

Figure 6 shows a plot of H_R vs P for $W = 10^9$ cycles per second and for several different values of center frequency ν. The expected center frequency range for laser type devices is from about 10^{12} cps to 10^{16} cps, and this range of frequency has been shown on the graph.

It is important to remember that the entropy rate H_R is actually the channel capacity of a noiseless light channel quantized as in our model in accordance with the fact that electromagnetic energy is not continuously variable but exists in discrete quanta. If there were no uncertainty in a quantized signal then H_R would be the channel capacity. H_R does not include the effect of quantum noise which stems from the uncertainty principle and is, therefore, only an upper limit to the actual channel capacity.

Channel Capacity Lower Bound for Large d

In this section we will derive a lower bound for the channel capacity. This bound is valid for large d only, although this restriction to $d \gg 1$ will be discussed after the bound has been derived.

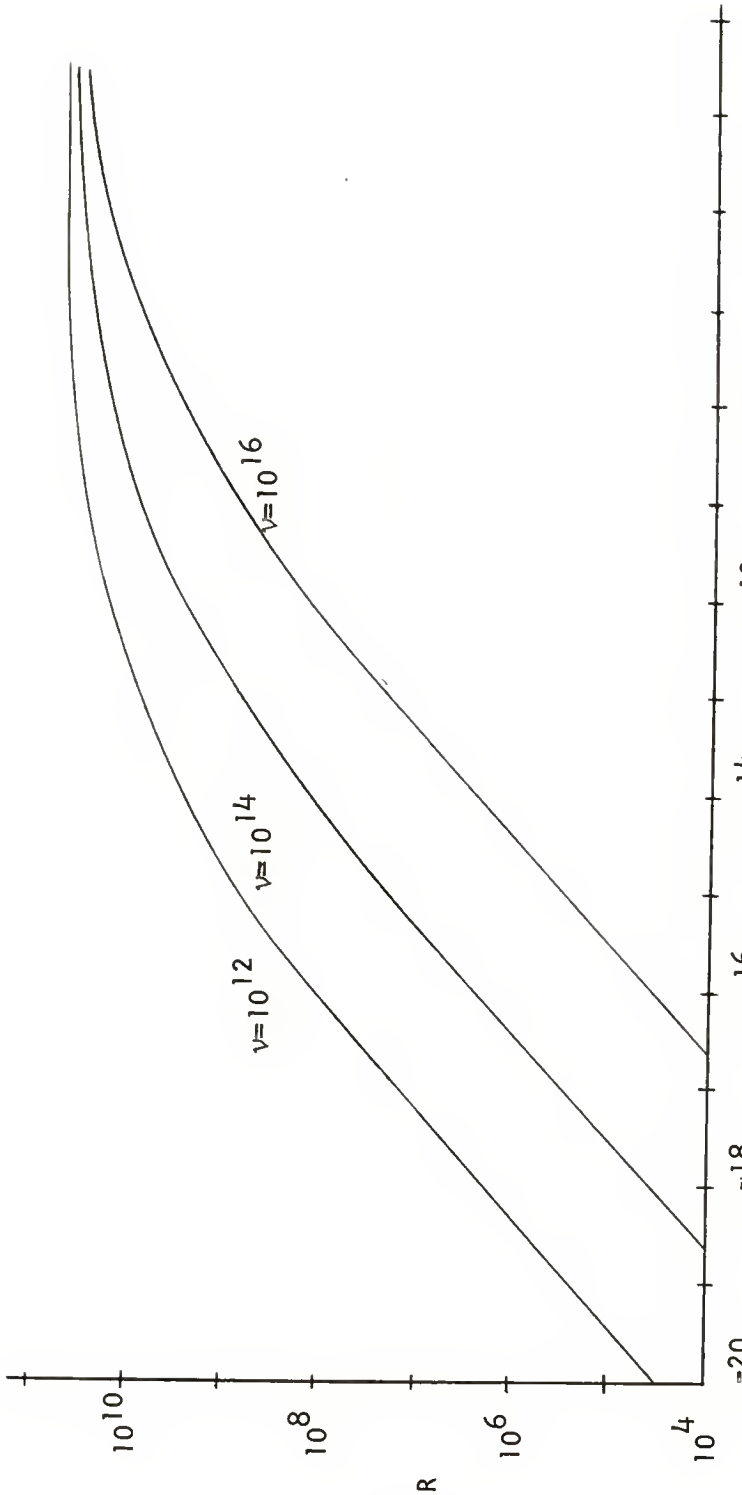

FIGURE 6 -- Upper Bound on Channel Capacity H_R in Bits per Second vs. the Received Power P in Watts for Several Center Frequencies at a Bandwidth of 109 cps

The derivation is based on the inequality

$$\text{(4.32)} \qquad \text{Max}_{p(n)} \left[H(n) - H_m(n) \right] \geq \text{Max}_{p(n)} H(n) - \text{Max}_{p(n)} H_m(n) ,$$

the left-hand side of which is part of equation (4.10). The $\text{Max}_{p(n)}H(n)$ has already been found to be the H of equation (4.15). We find the lower bound by over bounding $\text{Max}_{p(n)}H_m(n)$ which preserves the inequality in (4.23).

$$\text{(4.24)} \qquad H_m(n) = \sum_m p(m)\ H(n|m) ,$$

where $H(n|m)$ is the entropy of n, the signal plus noise, when the signal m is given, which is the sum of the entropies of n_i given m_i, namely,

$$\text{(4.25)} \qquad H(n|m) = \sum_{i=0}^{2TW} H(n_i|m_i).$$

The $H(n_i|m_i)$ are defined as

$$\text{(4.26)} \qquad H(n_i|m_i) = \sum_{n_i} p(n_i|m_i)\ \log\ p(n_i|m_i).$$

Since $p(n_i|m_i)$ is a Poisson function with mean m_i, it has variance m_i, also. The form of $p(n_i|m_i)$ which maximizes (4.26) subject to the constraint that the variance is fixed at m_i, is Gaussian [17]. Letting $p(n_i|m_i)$ be a Gaussian density with variance m_i in equation (4.26) gives $H(n_i|m_i) = \log \sqrt{2\pi e m_i}$. We know that the $H(n_i|m_i)$ of (4.26) can be no greater than this. Therefore,

$$(4.27) \qquad H(n_i|m_i) \leq \log \sqrt{2\pi e m_i} \ .$$

Putting this into (4.25) gives

$$(4.28) \quad H(n|m) \leq \sum_{i=0}^{2TW} \log \sqrt{2\pi e m_i} = \frac{1}{2} \log (2\pi e)^{2TW+1} m_0 m_1 \cdots m_{2TW} \ .$$

Substitution of (4.28) back into (4.24) results in the inequality

$$(4.29) \quad H_m(n) \leq \sum_{m} p(m) \frac{1}{2} \log (2\pi e)^{2TW+1} m_0 m_1 \cdots m_{2TW} \ ,$$

where $p(m)$ is the joint density function $p(m_0, m_1, \ldots, m_{2TW})$. Now since $\Sigma m_i = \bar{N}$, the right-hand side of (4.29) is a maximum if $p(m)$ is a $2TW + 1$ dimensional delta function at $m_0 = m_1 = \ldots = m_{2TW} = \frac{\bar{N}}{2TW + 1}$. Therefore,

$$\text{Max}_{p(n)} \ H_m(n) \leq \frac{1}{2} \log \left(2\pi e \frac{\bar{N}}{2TW + 1}\right)^{2TW+1},$$

or,

$$(4.30) \qquad \text{Max}_{p(n)} \ H_m(n) \leq \frac{2TW + 1}{2} \log \frac{2\pi e \bar{N}}{2TW+1} \ .$$

Finally, from equations (4.10) and (4.23) we derive our lower bound C_L as

$$C_L = H_R - \lim_{T \to \infty} \frac{1}{T} \left\{ \frac{2TW + 1}{2} \log \frac{2\pi e N}{2TW+1} \right\}$$

$$= H_R - W \log \frac{\pi e p}{W} \ .$$

From (4.19) and (4.20)

$$C_L = 2W \left[C(d) - \frac{1}{2} \log 2\pi ed \right] , \tag{4.31}$$

$$C_L = 2W \left[d \log\left(1 + \frac{1}{d}\right) + \log \left(\sqrt{d} + \frac{1}{\sqrt{d}}\right) - \frac{1}{2} \log 2\pi e \right]. \tag{4.32}$$

The inequality (4.23) from which this lower bound is derived is the difference of two entropies as is usual in computations involving channel capacity. However, the $\text{Max}_{p(n)}\ H(n)$ is the entropy of a discrete system. It is the logarithm of U, the number of distinguishable ways to arrange N photons into 2TW + 1 degrees of freedom. From this was subtracted the entropy of a continuous system with a Gaussian distribution. The entropy of a continuous system is not, in general, comparable to the entropy of a discrete system. They are comparable, however, when d becomes large, because then the entropy of the discrete process approaches the entropy of its continuous counterpart. That the bound is invalid for small d is easily seen from equation (4.31). For $d < 1/2\pi e$, the right-hand term becomes positive, and C_L, the lower limit, becomes greater than the upper limit $H_R = 2WC(d)$. For d greater than about 100, C_L should be an accurate lower bound for the channel capacity.

Making the substitution $d = P/2Wh\nu$ into equation (4.32) results in an expression for C_L in terms of the received power P. Figure 7 is a plot of the upper bound H_R and the lower bound C_L for a bandwidth of 10^9 cycles per second at a center frequency of 10^{14} cycles per second. The plot cannot be extended to received power levels less than about 10^{-10}

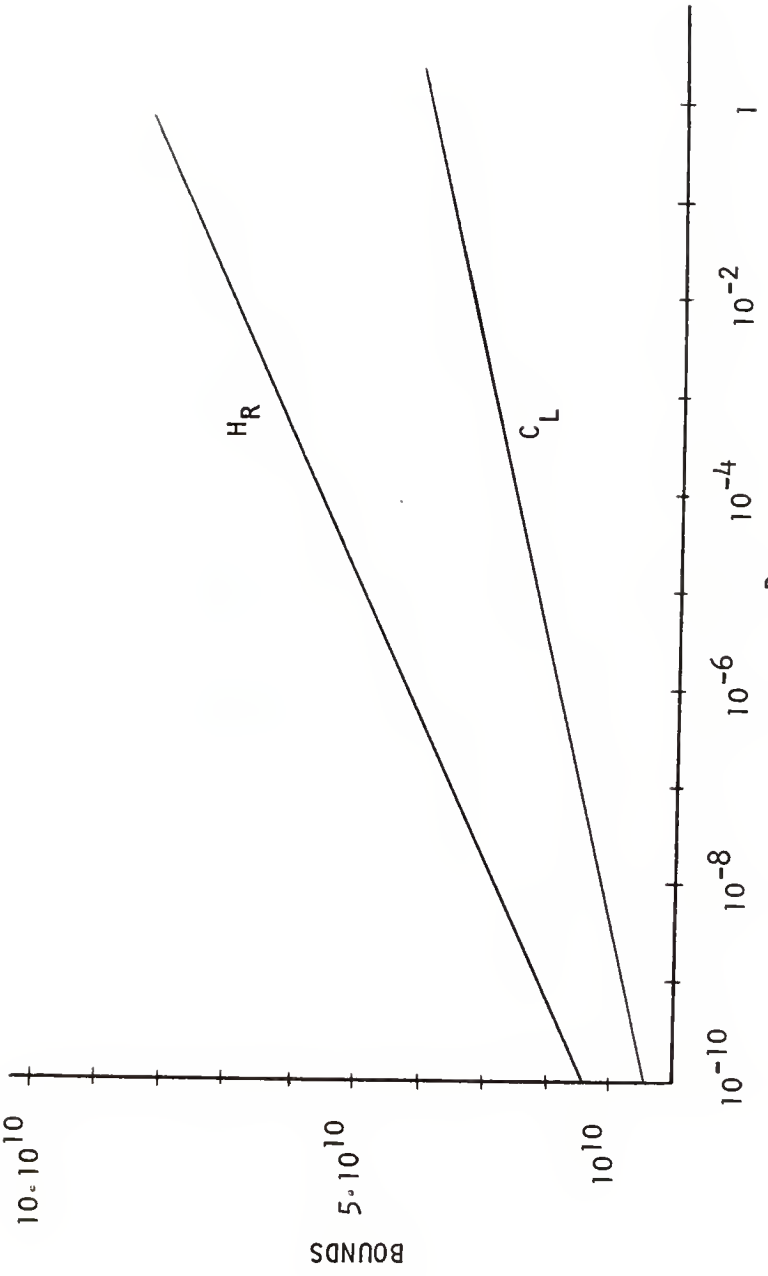

FIGURE 7 -- Channel Capacity Bounds H_R and C_L in Bits per Second vs. Received Power P in Watts for a Bandwidth of 10^9 cps and Center Frequency 10^{14} cps

watts, because d becomes too small and the lower bound C_L does not apply. In the case illustrated by the figure, when P is 10^{-10} watts, d is about 75.

Asymptotic Behavior of the Bounds

From (4.20) we see that

$$\lim_{d\to\infty} C(d) = \log e + \log d \quad . \tag{4.33}$$

Therefore,

$$\lim_{d\to\infty} H_R = 2W\ (\log e + \log d)\ , \tag{4.34}$$

and from (4.31)

$$\lim_{d\to\infty} C_L = 2W\ (\frac{1}{2} \log d + \frac{1}{2} \log \frac{e}{2\pi}). \tag{4.35}$$

As d gets larger, C_L approaches approximately 1/2 H_R. Thus, for large d our channel capacity is bound rather tightly by C_L and H_R.

An interesting form for C_L is found by substitution of equation (4.21) into (4.35). For large d this gives

$$C_L \simeq W \log \frac{P}{\frac{2\pi}{e}\, 2Wh\nu} \quad . \tag{4.36}$$

Now, since P is the total power of signal plus quantum noise, i.e., P = S + N, we can write (4.36) as

(4.37) $$C_L \simeq W \log \frac{S + N}{N} ,$$

where $N = (4\pi/e)Wh\nu$. This, of course, is in the form of Shannon's formula where the quantum noise is represented by additive white noise independent of the signal.

CHAPTER 5

THE INHERENT AMBIGUITY OF SIGNALS

The study of signals from a quantum mechanical point of view raises some questions about exactly what signals are and how they should be represented. This chapter, an appendage to the main body of this study, presents a brief discussion of these questions in very general terms.

Most previous work in communications theory is based on the concept of a precise continuum of possible signals. That is, that any signal can be represented by a precise function of time $s(t)$, and that $s(t)$ has a completely defined value for all values of the argument t. This is a very useful mathematical model, for it allows us to set up a one-to-one mapping between the set of all signals and a certain class of precise mathematical functions. This mapping is implicit in the statement: Let the signal be $s(t)$. We operate with these functions in a rigorous mathematical way and determine laws and theorems which apply to them. We then intuit from the mapping process that these laws also apply to the signals themselves.

Quantum theory tells us that signals do not exist as precise mathematical functions of time. It is tempting to preserve the mathematical model by asserting that signals do exist as precise mathematical functions, and that the difficulty arises in our inability to measure them accurately. This is certainly not the operational approach. Signals,

as we know them, are inherently uncertain. These general statements apply to any kind of time varying signals existing in the real world including the electromagnetic signals in which we are most interested.

Physical signals do not exist as precise mathematical functions of time, and there may be a certain danger in treating them as though they do. Slepian recognized this danger and discussed it briefly in [18]. It is not possible to put a mathematical function precisely into the form of a physically occurring signal because of the quantization of physical phenomena. Further, the measurement of any physical signal is subject to an irreducible uncertainty.

One attempt to circumvent this problem would be to assert that there is no such thing as a noiseless signal and that every signal must be represented as $s(t) + n(t)$, where $n(t)$ represents a noise which is always present. Such an assertion implies however, that the mathematical function $s(t) + n(t)$ precisely describes a naturally occurring phenomena. Such cannot be the case.

Obviously it is very useful to represent a signal s by a function of time $s(t)$, but what does $s(t)$ really mean? Suppose our mathematical function $s(t)$ has a value v_1 at t_1, i.e., $s(t_1) = v_1$. If we measure the actual signal at t_1 as accurately as is physically possible, we would get some number v. Quantum mechanics asserts that v is a random variable. If a number of identical signals could be prepared and measured at time t_1, then the results of these measurements would not be identical. The statement $s(t_1) = v_1$ then should mean only that the expected value of the measurement of the signal at t_1 is v_1. Thus, $s(t)$ should be interpreted as the expected value of the actual signal at time t. Obviously,

$s(t_1)$ approximates the actual signal at $t = t_1$ very well if the natural processes are such that the variance of v is small. Similarly, s(t) may be a very good approximation of the signal. And when s(t) is a good approximation, then valuable results are obtained through the mathematical analysis of continuous functions. The implication here is that we may not always be justified in assuming that signals are continuous and precise and that such an assumption may lead to erroneous conclusions. The theory of communications based on precise continuous signals applies to signals macroscopically observed but may fail to explain the true nature of signals when observed in minute detail.

CHAPTER 6

SUMMARY OF CONCLUSIONS

The purpose of this study has been to construct a reasonable model for a quantum communications channel and to study its properties. In the model proposed, every signal which is bandlimited to a bandwidth W and approximately timelimited to the time interval T has a wave function $\Psi(t)$ which can be expressed as

$$\Psi(t) = \sum_{i=0}^{2TW} a_i \Psi_i(t) \ , \tag{2.8}$$

where the a_i are complex constants and the $\Psi_i(t)$ are prolate spheroidal wave functions. An observable quantity of such a signal is represented by the Hermitian operator R where

$$R = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin\ \Omega(t - s)}{\pi(t - s)} \{\ \}\ ds \ . \tag{3.2}$$

If a measurement of the observable R is performed on one received photon of a signal with wave function $\Psi(t)$, then the outcome of the measurement will be one of the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{2TW}$. The probability that

the measurement is λ_i is A_i^2 for $i = 0, 1, \ldots, 2TW$, where A_i is the absolute magnitude of a_i. If the measurement is performed on N received photons, then the number of times λ_i is observed has the expectation value $N A_i^2$ for $i = 0, 1, \ldots, 2TW$.

Every wave function is associated with an ordered set of positive real numbers which can be written in vector form as $A = (A_0, A_1, \ldots, A_{2TW})$, where

$$\sum_{i=0}^{2TW} A_i^2 = 1 \quad . \tag{4.1}$$

Any A can represent a message. In order to transmit a message, the transmitter transmits an electromagnetic signal whose wave function corresponds to this A. The receiver performs the measurement represented by R on each of the received photons and, from the results of this observation, attempts to determine which wave function and therefore which message was transmitted.

It is shown in this study how the description of this model can be simplified as follows. The transmitter transmits the signal $m = (m_0, m_1, \ldots, m_{2TW})$ where

$$\sum_{i=0}^{2TW} m_i = \bar{N} \ , \tag{6.1}$$

and $\bar{N}$ is a constant. The received signal is $n = (n_0, n_1, \ldots, n_{2TW})$, where $n = m + q$, and $q = (q_0, q_1, \ldots, q_{2TW})$ is noise.

If there is no external or thermal noise, then q represents only quantum noise. In this case it is shown that each n_i of the received signal is a random variable with a Poisson distribution with mean m_i. An upper

bound on the channel capacity for a narrow band noiseless channel of bandwidth W, center frequency ν and average power P is found to be

$$H_R = 2W \left[d \log\left(1 + \frac{1}{d}\right) + \log(d + 1) \right] , \tag{6.2}$$

where $d = P/2Wh\nu$. The quantity d is recognized to be the average number of photons received per second per degree of freedom, and h is Planck's constant. A lower bound for this channel capacity which applies when d is large is

$$C_L = 2W \left[d \log\left(1 + \frac{1}{d}\right) + \log\left(\sqrt{d} + \frac{1}{\sqrt{d}}\right) - \frac{1}{2} \log 2\pi e \right] . \tag{4.32}$$

The channel capacity for a channel in which there is no external or thermal noise lies somewhere between the two bounds given by (4.32) and (6.2). Therefore, if the model presented in this study represents electromagnetic communications, it will never be possible to exceed the bound given by (6.2), but it may be possible under ideal conditions to attain the capacity given by (4.32).

APPENDIX A

PROLATE SPHEROIDAL WAVE FUNCTIONS

The purpose of this appendix is to summarize briefly the origin and characteristics of the prolate spheroidal wave functions of Slepian, Landau, and Pollak [4, 5, 6].

The set of prolate spheroidal wave functions are a countably infinite, complete set of real functions $\Psi_0(t)$, $\Psi_1(t)$, ... of the real variable t, each of whose Fourier transform is zero outside the interval $(-\Omega,\Omega)$ and which satisfy the following three equations for $T > 0$ and $\Omega > 0$:

$$\int_{-\infty}^{\infty} \Psi_i(t)\, \Psi_j(t)\, dt = \delta_{ij} \,, \tag{A-1}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_i(t)\, \Psi_j(t)\, dt = \lambda_i \delta_{ij} \,, \tag{A-2}$$

$$\lambda_i \Psi_i(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi(t - s)}\, \Psi_i(s)\, ds \,. \tag{A-3}$$

The quantity δ_{ij} is the Kronecker delta function. The $\Psi_i(t)$ and λ_i are the eigenfunctions and eigenvalues, respectively, of equation (A-3) from which they are derived. The eigenvalues are real, non-degenerate, and are ordered as $\lambda_0 > \lambda_1 > \lambda_2 \ldots$. The λ_i are always less than one. For

given values of Ω and T the λ_i fall off rapidly with increasing i once i has exceeded $\Omega T/\pi$. The function $\Psi_i(t)$ is either even or odd according as i is even or odd and has exactly i zeros in the interval (-T/2, T/2).

From (A-3) we see that the $\Psi_i(t)$ are functions of Ω, T and t. In most cases of interest, however, the Ω and T are constants and t is the only variable.

Slepian and Pollak arrive at equation (A-3) by asking the following question: What function loses the least amount of its energy when first timelimited and then bandlimited? The answer to this question turns out to be the timelimited version of $\Psi_0(t)$, where $\Psi_0(t)$ is defined as the eigenfunction belonging to the largest eigenvalue of equation (A-3). This function was discovered earlier by Chalk [19].

The functions $\Psi_i(t)$ are scaled versions of certain of the angular prolate spheroidal wave functions discovered and tabulated earlier [20, 21]. Just as Bessel functions can be found by the solution of the wave equation in cylindrical coordinates, the angular prolate spheroidal wave functions were found originally by solving the wave equation in prolate spheroidal coordinates. In what follows we use the notation of Flammer [20] and show briefly how the functions $\Psi_i(t)$ can be derived from his work.

The prolate and oblate coordinate systems are two of the eleven coordinate systems in which the scalar wave equation

$$(\nabla^2 + k^2)\, f = 0 \qquad \text{(A-4)}$$

is separable. If the wave equation is expressed in terms of prolate spheroidal coordinates, then the solution can be expressed in terms of the Lamé product, as in Flammer's equation (2.2.5),

$$f_{mn} = S_{mn}(C,\eta)\ R_{mn}(C,\xi) \begin{matrix}\cos\\ \sin\end{matrix} m\,\Phi, \tag{A-5}$$

where η,ξ,Φ designate the prolate spheroidal coordinates and C is a real constant proportional to k. The functions $S_{mn}(C,\eta)$ are called angular prolate spheroidal wave functions and are solutions of Flammer's equation (2.2.6), which is

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(C,\eta)\right] + \left[\lambda_{mn} - C^2\eta^2 - \frac{m^2}{1-\eta^2}\right] S_{mn}(C,\eta) = 0. \tag{A-6}$$

Solutions of this equation exist only for certain values of the separation constant λ_{mn}. For $m = 0$ and $C^2 = 0$, equation (A-6) reduces to a form of Legendre's equation whose solutions are Legendre's polynomials. Therefore, the solutions $S_{mn}(C,\eta)$ are called prolate spheroidal wave functions only for $C^2 \neq 0$.

The functions $R_{mn}(C,\xi)$ are called radial prolate spheroidal wave functions and are the solutions to an equation similar to (A-6). Flammer shows that $S_{mn}(C,\eta)$ and $R_{mn}(C,\xi)$ satisfy several different integral equations.

These functions are of interest to us primarily because they satisfy the integral equation

$$(A\text{-}7) \qquad S_{mn}(C,\eta)\, R^{(1)}_{mn}(C,\xi) \cos m\phi =$$

$$(4\pi j^n)^{-1} \int_0^{2\pi} \int_{-1}^{1} e^{jy}\, S_{mn}(C,\eta') \cos m\phi'\, d\eta'\, d\phi' ,$$

where

$$y = C\,[\eta'\eta\xi + (1-\eta'^2)^{1/2}\,(1-\eta^2)^{1/2}\,(\xi^2-1)^{1/2} \cos(\phi-\phi')] .$$

These are Flammer's equations (7.1.1) and (7.1.2a). $R^{(1)}_{mn}(C,\xi)$ is called a radial prolate spheroidal wave function of the first kind. If we substitute $m = 0$ and $\xi = 1$ into (A-7) we get

$$(A\text{-}8) \qquad S_{on}(C,\eta)\, R^{(1)}_{on}(C,1) = (4\pi j^n)^{-1} \int_0^{2\pi} \int_{-1}^{1} e^{jC\eta'\eta}\, S_{on}(C,\eta')\, d\eta' d\phi' .$$

Making the change of variable from η to τ and from η' to s and performing the integration in ϕ' gives

$$(A\text{-}9) \qquad 2j^n\, R^{(1)}_{on}(C,1)\, S_{on}(C,\tau) = \int_{-1}^{1} e^{jC\tau s}\, S_{on}(C,s)\, ds .$$

This equation is valid for $n = 0, 1, 2 \ldots$, and is equation (25) in Slepian and Pollak [4]. The iterate of (A-9) is found by multiplying both sides by $e^{-jC\tau t}$ and integrating from $\tau = -1$ to $\tau = 1$. This is done as follows;

$$2j^n\, R^{(1)}_{on}(C,1) \int_{-1}^{1} e^{-jC\tau t}\, S_{on}(C,\tau)\, d\tau = \int_{-1}^{1} \int_{-1}^{1} e^{-jC\tau(s-t)}\, S_{on}(C,s)\, d\tau ds,$$

$$2j^n R_{on}^{(1)}(C,1)\,[2j^n R_{on}^{(1)}(C,1)\,S_{on}(C,-t)] = \int_{-1}^{1} S_{on}(C,s)\,\frac{2\sin C(s-t)}{C(s-t)}\,ds,$$

$$\text{(A-10)}\qquad 2j^{2n}\,[R_{on}^{(1)}(C,1)]^2\,S_{on}(C,-t) = \int_{-1}^{1} \frac{\sin C(t-s)}{C(t-s)}\,S_{on}(C,s)\,ds\,.$$

Examination of the series expansion of $S_{on}(C,t)$, Flammer's equation (3.4.1), shows that $S_{on}(C,t)$ is even or odd when n is even or odd, respectively. This means that $j^{2n}S_{on}(C,-t) = S_{on}(C,t)$, and we can write (A-10) as

$$\text{(A-11)}\qquad \frac{2C}{\pi}[\,R_{on}^{(1)}(C,1)]^2\,S_{on}(C,t) = \int_{-1}^{1} \frac{\sin C(t-s)}{\pi(t-s)}\,S_{on}(C,s)\,ds\,,$$

which is equation (24) of Slepian and Pollak [4].

Finally, defining

$$\text{(A-12)}\qquad \lambda_n(C) = \frac{2C}{\pi}\,[\,R_{on}^{(1)}(C,1)]^2$$

and

$$\text{(A-13)}\qquad \Psi_n(C,t) = \sqrt{\frac{2\lambda_n(C)}{T\int_{-1}^{1}[S_{on}(C,t)]^2\,dt}}\;S_{on}(C,\tfrac{2t}{T}),$$

and making the appropriate change of variables converts (A-9) and (A-11) into

$$\text{(A-14)}\qquad \frac{j^n\Omega\;R_{on}^{(1)}(C,1)}{\pi}\,\Psi_n(C,t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{j\omega t}\,\Psi_n(C,\tfrac{\omega T}{2\Omega})\,d\omega$$

and

$$\lambda_n \Psi_n(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi(t - s)} \Psi_n(s)\, ds \,. \tag{A-3}$$

The orthogonality of the set $\Psi_0(t)$, $\Psi_1(t)$, ... over the two different time intervals $(-\infty,\infty)$ and $(-T/2, T/2)$ as stated in equations (A-1) and (A-2) is proved from equation (A-3) in Slepian and Pollak [4].

The utility of the functions $\Psi_i(t)$ is embodied in the fact that they are the eigenvalues of the finite Fourier transform kernel as illustrated by equation (A-14). It has long been recognized that the eigenfunctions and eigenvalues of (A-3) are important in applied mathematics [19, 22]. The fact that the $\Psi_i(t)$ were first derived from the solutions of the wave equation in prolate spheroidal coordinates seems only incidental. It may be unfortunate that these functions have been given the frighteningly restrictive name of prolate spheroidal wave functions. It is evident that these functions have great applicability in the field of communications theory and in other fields where prolate spheroids are conspicuously absent. Their use in antenna theory itself, the field of study which led to their development, is even broader than their name suggests [23].

APPENDIX B

EXTENSION TO THE NARROW BAND CASE

Most of the previous analytical work in the body of this thesis has dealt with video type functions, i.e., functions whose spectrum $a(\omega)$ exists only in one interval centered on $\omega = 0$, or other functions derived directly from these. It is the purpose of this appendix to show that the results found in Chapter 4 for the video case are valid also for the bandpass case.

The prolate spheroidal wave functions are all limited to the bandwidth $(-\Omega,\Omega)$. We have shown in equation (3.10) that a function whose spectrum is zero outside the interval $(0,\Omega)$ can be represented as a sum of prolate spheroidal wave functions and that such a function is complex. Thus, the function $\Psi(t)$ whose spectrum $a(\omega)$ is zero outside $(0,\Omega)$ can be represented by the equation

$$\Psi(t) = \Phi(t) + j\,\hat{\Phi}(t)\ , \tag{3.4}$$

where $\Phi(t)$ is a real function whose spectrum is zero outside $(-\Omega,\Omega)$ and $\hat{\Phi}(t)$ is the Hilbert transform of $\Phi(t)$. Of course, the spectra of $\Phi(t)$ and $j\,\hat{\Phi}(t)$ combine in such a way that the spectrum of $\Phi(t) + j\,\hat{\Phi}(t)$ exists only in the interval $(0,\Omega)$.

It will be remembered that we chose to use functions with one-sided spectra, like $\Psi(t)$ in (3.4), primarily because of the quantum

mechanical interpretation of $|a(\omega)|^2$ as a probabilty density. This conventional approach led us into complex wave functions whose real and imaginary parts were Hilbert transforms. We could have used double-sided spectra with the interpretation that $2|a(\omega)|^2$ was a probabilty density function for positive ω only. Using this approach, we could have simply ignored $|a(\omega)|^2$ for negative ω, and this approach would have led us to exactly the same conclusions and results as the scheme used. Of course, we would have specified that the two sides of the spectrum were complex conjugates, and real wave functions would have resulted.

In what follows we show that a function $\Psi(t)$ whose Fourier transform $a(\omega)$ exists only in the interval $(\omega_0, \omega_0 + \Omega)$ can be represented in terms of a sum of 2TW + 1 functions of the form $\Psi_i(t)e^{j\omega 0t}$ where the $\Psi_i(t)$ are the PSW functions. The relationship between $\Psi(t)$ and $a(\omega)$ is

$$\Psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{\omega_0}^{\omega_0+\Omega} a(\omega)e^{j\omega t}\,d\omega \,. \tag{B-1}$$

Making a change of variable to $\omega' = \omega - \omega_0$ results in

$$\Psi(t) = \int_0^{\Omega} a_v(\omega')e^{j\omega' t}\,d\omega'\; e^{j\omega_0 t} \,, \tag{B-2}$$

where $a_v(\omega)$ is the video version of $a(\omega)$, i.e., $a_v(\omega) = a(\omega + \omega_0)$.

Using the same reasoning which resulted in (3.4), we can express $\Psi(t)$ as

(B-3) $$\Psi(t) = \{\Phi(t) + j\,\hat{\Phi}(t)\}\, e^{j\omega_0 t} \quad ,$$

where $\Phi(t)$ is a real function whose Fourier transform $\alpha(\omega)$ is defined by

(3.5) $$\begin{aligned} \alpha(\omega) &= 1/2\ a_V(\omega) && \omega > 0 \\ &= 0 && \omega = 0 \\ &= 1/2\ a_V^*(-\omega) && \omega < 0 \quad . \end{aligned}$$

Thus, the spectrum of $\Phi(t)$ is zero outside of the interval $(-\Omega,\Omega)$, and $\Phi(t)$ can, therefore, be expressed as a sum of $2TW + 1$ prolate spheroidal functions with real coefficients. As before, $\hat{\Phi}(t)$ denotes the Hilbert transform of $\Phi(t)$. The spectrum of $\hat{\Phi}(t)$ is

(3.6) $$\begin{aligned} \alpha_1(\omega) &= -\ 1/2\ j a_V(\omega) && \omega > 0 \\ &= 0 && \omega = 0 \\ &= 1/2\ j a_V^*(-\omega) && \omega < 0 \quad . \end{aligned}$$

Once the function $\Phi(t)$ is known, then $\hat{\Phi}(t)$ can be found by the relationship (3.6). Since the spectrum of $\hat{\Phi}(t)$ is zero outside of $(-\Omega,\Omega)$, it can also be expressed as a sum of prolate spheroidal wave functions. Therefore, equation (B-3) can be written as:

(B-4) $$\Psi(t) = \{\sum_{i=0}^{\infty} A_i e^{j\theta_i}\, \Psi_i(t)\}\, e^{j\omega_0 t} \quad ,$$

or

(B-5) $$\Psi(t) = \sum_{i=0}^{\infty} a_i \Psi_i(t)\, e^{j\omega_0 t} \quad .$$

This shows that a $\Psi(t)$ whose spectrum is zero outside of the interval $(\omega_0, \omega_0 + \Omega)$ can be represented as a sum of functions of the form $\Psi_i(t)e^{j\omega_0 t}$. All of the analyses for the video functions treated in the main body of this thesis are based on the fact that the prolate spheroidal wave functions satisfied equations (A-1), (A-2), and (A-3). The functions $\Psi_i(t)e^{j\omega_0 t}$ satisfy a similar set of equations and, therefore, all analyses and conclusions which apply to video signals apply also to bandpass functions whose spectrum is zero outside of any continuous interval of length Ω. These important equations for the functions $\Psi_i(t)e^{j\omega_0 t}$ are

$$\int_{-\infty}^{\infty} (\Psi_i(t)\ e^{j\omega_0 t})(\Psi_j(t)e^{j\omega_0 t})^*\ dt = \delta_{ij}\ , \tag{B-6}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} (\Psi_i(t)e^{j\omega_0 t})(\Psi_j(t)e^{j\omega_0 t})^*\ dt = \lambda_i\delta_{ij}\ , \tag{B-7}$$

$$\lambda_i(\Psi_i(t)e^{j\omega_0 t}) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi(t - s)}\ e^{\ j\omega_0(t-s)}(\Psi_i(s)e^{j\omega_0 s})\ ds\ . \tag{B-8}$$

As usual, the asterisk denotes the complex conjugate. These equations are easily verified from equations (A-1) through (A-3). The observable R is represented by the operator

$$R = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - s)}{\pi(t - s)}\ e^{j\omega_0(t-s)}\{\ \}\ ds. \tag{B-9}$$

Expressed in matrix form R is given by equation (3.16) for both the video and bandpass cases.

Furthermore, substitution of the functions $\Psi_i(t)e^{j\omega_0 t}$ for $\Psi_i(t)$ in equation (2.6) shows that the theorem of Landau and Pollak stated in Chapter 2 applies to these bandpass functions as well as the video functions. Therefore, if the spectrum of a function is limited to a continuous interval $\Omega = 2\pi W$, and if the function is approximately timelimited in the sense of equations (2.6) and (2.7), then the function can be represented in terms of $2TW + 1$ functions $\Psi_i(t)e^{j\omega_0 t}$ as

$$\Psi(t) = \sum_{i=0}^{2TW} a_i \Psi_i(t)\, e^{j\omega_0 t} \quad . \tag{B-10}$$

We have shown, then, that bandpass functions can be treated exactly as the video functions, and therefore, the procedures and conclusions of the body of this paper apply equally well to the bandpass case.

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BIOGRAPHY

John Benjamin O'Neal, Jr., was born in Macon, Georgia, on October 15, 1934. He was educated in the public school system of Columbia, South Carolina, and was graduated from Dreher High School in June, 1954. He attended the Georgia Institute of Technology from June, 1954, until June, 1957, as a cooperative student. During this period he worked part-time as a student engineer for the Southern Bell Telephone and Telegraph Company in their equipment engineering department at Columbia, South Carolina. He was graduated from the Georgia Institute of Technology in June, 1957, with the degree of Bachelor of Electrical Engineering. He attended the University of South Carolina from September, 1958, until September, 1959, and was awarded the degree of Master of Engineering in June, 1960. From September, 1959, until September, 1960, he was employed by the Martin Company in Orlando, Florida, and did communications systems analysis work there. From September, 1960, until the present time he has pursued his work toward the degree of Doctor of Philosophy at the University of Florida while being employed as a teaching assistant and research assistant by the Department of Electrical Engineering. In the summer of 1962 he was employed by Autonetics Inc., in Anaheim, California.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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